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Extremal Problems for Hyperbolic Systems with Boundary Conditions Involving Integral Time Lags

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Abstract: Extremal problems for integral time lag hyperbolic systems are presented. The optimal boundary control problems for hyperbolic systems in which integral time lags appear in the Neumann boundary conditions are solved. Such systems constitute, in a linear approximation, a universal mathematical model for many processes in which transmission signals at a certain distance with electric, hydraulic and other long lines take place. The time horizon is fixed. Making use of Dubovicki-Milyutin scheme, necessary and sufficient conditions of optimality for the Neumann problem with the quadratic performance functionals and constrained control are derived.

Keywords: boundary control, hyperbolic systems, Neumann boundary conditions, integral time lags

1. Introduction

Igor V. Girsanov, was one of the first mathematicians to study general extremum problems and to realize the feasibility and desirability of a unified theory of extremal problems, based on a functional–analytic approach. His book [2] was apparently the first systematic exposition of a unified approach to the theory of extremal problems. This approach was based on the ideas of Dubovicki and Milyutin concerning extremum problems in the presence of constraints. Dubovicki and Milyutin found a necessary condition for an extremum in the form of an equation set down in the language of functional analysis.

For instance, in the paper [3], the Dubovicki-Milyutin method was applied for solving optimal control problems for parabolic-hyperbolic systems. The existence and uniqueness of solutions of such parabolic-hyperbolic systems with the Dirichlet boundary conditions are discussed. Making use of the Dubovicki-Milyutin method necessary and sufficient conditions of optimality for the Dirichlet problem with the quadratic performance functional and constrained control are derived.

In the papers [4–9], the Dubovicki-Milyutin method was applied for solving boundary optimal control problems for the case of time lag parabolic equations [4] and for the case of parabolic equations involving time-varying lags [5–7], multiple time-varying lags [8], and integral time lags [9] respectively. Sufficient conditions for the existence of a unique solution of such parabolic equations [4–9] are presented.

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Consequently, in the papers [4–9], the linear quadratic problems of parabolic systems with time lags given in various forms (constant time lags [4], time-varying lags [5–7], multiple time-varying lags [8], integral time lags [9] etc.) were solved.

In the papers [12–15] the linear quadratic problems of optimal boundary control for hyperbolic systems with constant time delays [12], multiple constant time delays [13], time-varying delays [14] and multiple time-varying delays [15] are investigated.

Sufficient conditions for the existence of a unique solution of such hyperbolic equations with the Neumann boundary conditions [12–15] are presented. Making use of Dubovicki-Milyutin method [6], necessary and sufficient conditions of optimality with the quadratic cost functions and constrained boundary control are derived for the Neumann problem.

Extremal problems for integral time lag hyperbolic systems are investigated. The purpose of this paper is to show the use of Dubovicki-Milyutin theorem [6] in solving optimal control problems for hyperbolic systems.

As an example, an optimal boundary control problem for a system described by a linear partial differential equation of hyperbolic type in which integral time lags appear in the Neumann boundary condition is considered.

Equations (1)–(5) constitute, in a linear approximation, a universal mathematical model for many processes in which transmission signals at a certain distance with electric, hydraulic and other long lines take place.

In the processes mentioned above time-delayed feedback signals are introduced at the boundary of a system's spatial domain. Then the signal at the boundary of a system's spatial domain at any time depends on the signal emitted earlier. This leads to the boundary conditions involving integral time lags.

Sufficient conditions for the existence of a unique solution of such hyperbolic equation with the Neumann boundary condition are presented.

The performance functionals have the quadratic form. The time horizon is fixed. Finally, we impose some constraints on the boundary control. Making use of the Dubovicki-Milyutin theorem [6], necessary and sufficient conditions of optimality with the quadratic performance functionals and constrained control are derived for the Neumann problem.

2. Preliminaries

Consider now the distributed-parameter system described by the following hyperbolic equation

$$\frac{\partial^2 y}{\partial t^2} + A(t)y = f \qquad x \in \Omega, \quad t \in (0,T) \tag{1}$$

$$y(x,0) = y_1(x) \qquad x \in \Omega \tag{2}$$

$$\frac{\partial y(x,0)}{\partial t} = y_2(x) \qquad x \in \Omega \tag{3}$$

$$\frac{\partial y}{\partial \eta_A} = \int_a^b y(x,t-h)\,dh + Gv \qquad x \in \Gamma, \quad t \in (0,T) \tag{4}$$

$$y(x,t') = \Psi_0(x,t') \qquad x \in \Gamma, \quad t' \in [-b,0) \tag{5}$$

where: $\Omega \in \mathrm{R}^n$ – a bounded, open set with boundary Γ which is a C^∞ – manifold of dimension (n − 1). Locally, Ω is totally on one side of Γ.

$$y \equiv y(x,t;v), \quad f \equiv f(x,t), \quad v \equiv v(x,t),$$

$$Q = \Omega \times (0,T), \quad \bar{Q} = \Omega \times [0,T],$$

$$\Sigma = \Gamma \times (0,T), \quad \Sigma_0 = \Gamma \times [-b,0)$$

h is a time lag such that $h \in (a,b)$ and $a > 0$, Ψ_0 is an initial function defined on Σ_0, G is a linear continuous operator on $L^2(\Sigma)$ into

$\left(H^{5/2}\Xi^{5/2}(\Sigma)\right)'$ with $v \in L^2(\Sigma)$ and $Gv \in H^{-5/2}\Xi^{-5/2}(\Sigma)$.

The hyperbolic operator $\frac{\partial^2 y}{\partial t^2} + A(t)$ in the state equation (1) satisfies the hypothesis of Section 1, Chapter 4 ([17], Vol. 2, p. 2) and $A(t)$ is given by

$$A(t)y = -\sum_{i,j=1}^{n} \frac{\partial}{\partial x_i}\left(a_{ij}(x,t)\frac{\partial y(x,t)}{\partial x_j}\right) \tag{6}$$

and the functions $a_{ij}(x,t)$ satisfy the condition

$$\sum_{i,j=1}^{n} a_{ij}(x,t)\varphi_i\varphi_j \ge \alpha \sum_{i=1}^{n}\varphi_i^2,$$

$$\alpha > 0, \quad \forall (x,t) \in \bar{Q}, \quad \forall \varphi_i \in \mathrm{R}, \quad a_{ij} = a_{ji} \forall i,j \tag{7}$$

where $a_{ij}(x,t)$ are real C^∞ functions defined on $\bar{Q}$ (closure of Q).

The equations (1)–(5) constitute a Neumann problem. Then the left-hand side of (4) is written in the form

$$\frac{\partial y}{\partial \eta_A} = \sum_{i,j=1}^{n} a_{ij}(x,t)\cos(n,x_i)\frac{\partial y(x,t)}{\partial x_j} = q(x,t) \tag{8}$$

where $\frac{\partial}{\partial \eta_A}$ is a normal derivative at Γ, directed towards the exterior of Ω, $\cos(n, x_i)$ is an i-th direction cosine of n, n-being the normal at Γ exterior to Ω and

$$q(x,t) = \int_a^b y(x,t-h)\,dh + Gv(x,t) \tag{9}$$

First we shall prove the existence of a unique solution of the mixed initial-boundary value problem (1)–(5) defined by transposition, i.e.

$$\langle y, u'' + Au\rangle = L(u) \qquad \forall u \in X^1(Q) \tag{10}$$

where

$$L(u) = \langle f,u\rangle + \langle q,u\rangle + \langle y_2, u(0)\rangle - \langle y_1, u'(0)\rangle \tag{11}$$

and we denote by $X^1(Q)$ the space described by the solutions u of the following adjoint problem

$$\left.\begin{aligned}
&u'' + Au = \Phi && x \in \Omega, \quad t \in (0,T),\\
&u(x,T) = 0 && x \in \Omega,\\
&u'(x,T) = 0 && x \in \Omega,\\
&\frac{\partial u}{\partial \eta_A} = 0 && x \in \Gamma, \quad t \in (0,T).
\end{aligned}\right\} \tag{12}$$

where: $\Phi \in H^{1,2}_{0,0}(Q)$ = closure of $\mathcal{D}(Q)$ in $H^{1,2}(Q)$.

For this purpose, we define the following space ([17], Vol. 2, Chapter 5, p. 131)

$$\mathcal{D}^{-1}_{A+\mathcal{D}_t^2}(Q) \overset{df}{=} \left\{ y \mid y \in H^{-1,-2}(Q),\ y'' + Ay \in \Xi^{-3,-3}(Q) \right\} \tag{13}$$

where the spaces $H^{-1,-2}(Q)$ and $\Xi^{-3,-3}(Q)$ are defined by (9.5) and (10.4) of Chapter 5 in ([17], Vol. 2) respectively. Under the norm of the graph $\mathcal{D}^{-1}_{A+\mathcal{D}_t^2}(Q)$ is a Hilbert space.

Then, the solution of (10) belongs to $\mathcal{D}^{-1}_{A+\mathcal{D}_t^2}(Q)$.

We shall restrict our considerations to the case where $v \in L^2(\Sigma)$. For simplicity, we shall introduce the following notations

$$E_j \overset{df}{=} \left((j-1)a, ja\right), \quad Q_j = \Omega \times E_j, \quad \Sigma_j = \Gamma \times E_j,$$

for j = 1, 2, …, K.

The existence of a unique solution of the mixed initial-boundary value problem (1)–(5) on the cylinder Q can be proved using a constructive method, i.e. by first solving problem (10) in the subcylinder Q_1, and in turn in Q_2 etc., until the procedure cover the whole cylinder Q. In this way the solution in the previous step determines the next one.

Consequently, using the Theorem 10.1 of [17] (Vol. 2, p. 132) we can prove the following result.

Theorem 1 *Let y_1, y_2, Ψ_0, v and f be given, with*

$$y_1 \in \Xi^{-3/2}(\Omega), \qquad y_2 \in \Xi^{-5/2}(\Omega),$$

$$\Psi_0 \in H^{-5/2}\Xi^{-5/2}(\Sigma_0), \qquad v \in L^2(\Sigma), \qquad f \in \Xi^{-3,-3}(Q).$$

Then, there exists a unique solution $y \in \mathcal{D}^{-1}_{A+\mathcal{D}_t^2}(Q)$ for the problem (1)–(5) defined by transposition (10). Moreover, $y(\bullet, ja) \in \Xi^{-3/2}(\Omega)$, and $y'(\bullet, ja) \in \Xi^{-5/2}(\Omega)$ for $j = 1, \ldots, K$. The proof of the Theorem 1 can be find in [11].

We refer to Lions and Magenes ([17], Vol. 2) for the definition and properties on $H^{r,s}(Q)$ and $(H^{r,s})'$ respectively. In the sequel, we shall fix $f \in \Xi^{-3,-3}(Q)$.

3. Problem Formulation. Optimization Theorems

In this paper we shall consider the optimal boundary control problem i.e. $v \in L^2(\Sigma)$.

Let us denote by $Y = \mathcal{D}^{-1}_{A+\mathcal{D}_t^2}(Q)$ the space of states and by $U = L^2(\Sigma)$ the space of controls. The time horizon T is fixed in our problem.

The performance functional is given by

$$I(v) = \lambda_1 \left\| y(v) - z_d \right\|^2_{H^{-1,-2}(Q)} + \lambda_2 \langle Nv, v \rangle_{L^2(\Sigma)} \tag{14}$$

where $\lambda_i \geq 0$ and $\lambda_1 + \lambda_2 > 0$, z_d is a given element in $H^{-1,-2}(Q)$, and N is a strictly positive linear operator on $L^2(\Sigma)$ into $L^2(\Sigma)$.

Finally, we assume the following constraints on the control:

$$v \in U_{ad} \tag{15}$$

where U_{ad} is a closed, convex set with non-empty interior, a subset of U.

Let $y(x, t, v)$ denote the solution of (1)–(5) at (x, t) corresponding to a given control $v \in U_{ad}$. We note from the Theorem 1 that for any $v \in U_{ad}$ the cost function (15) is well defined since

$$y \in \mathcal{D}^{-1}_{A+\mathcal{D}_t^2}(Q) \subset H^{-1,-2}(Q).$$

The optimal control problem (1)–(5), (14), (15) will be solved as the optimization one in which the function v is the unknown function. Making use of Dubovicki-Milyutin theorem [10] we shall derive the necessary and sufficient conditions of optimality for the optimization problem (1)–(5), (14), (15).

The solution of the stated optimal control problem is equivalent to seeking a pair $(y^0, v^0) \in E = \mathcal{D}^{-1}_{A+\mathcal{D}_t^2}(Q) \times L^2(\Sigma)$ which satisfies the equation (1)–(5) and minimizing the performance functional (14) with the constraints on the control (15).

Theorem 2 *The solution of the optimization problem (1)–(5), (14), (15) exists and it is unique with the assumptions mentioned above; the necessary and sufficient conditions of the optimality are characterized by the following system of partial differential equations and inequalities.*

State equation

$$\frac{\partial^2 y^0}{\partial t^2} + A(t) y^0 = f \qquad x \in \Omega, \quad t \in (0, T) \tag{16}$$

$$y^0(x, 0) = y_1(x) \qquad x \in \Omega \tag{17}$$

$$\frac{\partial y^0(x, 0)}{\partial t} = y_2(x) \qquad x \in \Omega \tag{18}$$

$$\frac{\partial y^0}{\partial \eta_A} = \int_a^b y^0(x, t-h)\,dh + G v^0 \qquad x \in \Gamma, \quad t \in (0, T) \tag{19}$$

$$y^0(x, t') = \Psi_0(x, t') \qquad x \in \Gamma, \quad t' \in [-b, 0) \tag{20}$$

Adjoint equations

$$\frac{\partial^2 p}{\partial t^2} + A(t) p = \lambda_1 \Lambda_1 (y^0 - z_d) \qquad x \in \Omega, \quad t \in (0, T) \tag{21}$$

$$\frac{\partial p}{\partial \eta_A} = \int_a^b p(x, t+h)\,dh \qquad x \in \Gamma, \quad t \in (0, T-b) \tag{22}$$

$$\frac{\partial p}{\partial \eta_A} = \int_a^{T-t} p(x, t+h)\,dh \qquad x \in \Gamma, \quad t \in (T-b, T-a) \tag{23}$$

$$\frac{\partial p}{\partial \eta_A} = 0 \qquad x \in \Gamma, \quad t \in (T-a, T) \tag{24}$$

$$p(x, T) = 0 \qquad x \in \Omega \tag{25}$$

$$\frac{\partial p(x, T)}{\partial t} = 0 \qquad x \in \Omega \tag{26}$$

where Λ_1 is a canonical isomorphism of $H^{-1,-2}(Q)$ onto $H^{1,2}_{0,0}(Q)$.

Maximum condition

$$\left\langle G^* p(v^0) + \lambda_2 N v^0, v - v^0 \right\rangle_{L^2(\Sigma)} \geq 0 \qquad \forall v \in U_{ad} \tag{27}$$

We can also notice that

$$\frac{\partial p}{\partial \eta_A} = \sum_{i,j=1}^{n} a_{ji}(x, t) \cos(n, x_i) \frac{\partial p}{\partial x_j} \tag{28}$$

OUTLINE OF THE PROOF:
According to the Dubovicki-Milyutin theorem [6], we approximate the set representing the inequality constraints by the regular admissible cone, the equality constraint by the regular tangent cone and the performance functional by the regular improvement cone.

a) Analysis of the equality constraint

The set Q_1 representing the equality constraint has the form

$$Q_1 = \left\{ \begin{array}{lll} \dfrac{\partial^2 y}{\partial t^2} + A(t) y = f & x \in \Omega, & t \in (0,T) \\ y(x,0) = y_1(x) & x \in \Omega & \\ \dfrac{\partial y(x,0)}{\partial t} = y_2(x) & x \in \Omega & \\ \dfrac{\partial y}{\partial \eta_A} = \int_a^b y(x,t-h)\,dh + Gv & x \in \Gamma, & t \in (0,T) \\ y(x,t') = \Psi_0(x,t') & x \in \Gamma, & t' \in [-b,0) \end{array} \right\} \quad (29)$$

We construct the regular tangent cone of the set Q_1 using the Lusternik theorem (Theorem 9.1 [2]). For this purpose, we define the operator P in the form

$$\begin{aligned} P(y,v) = \Bigg(& \frac{\partial^2 y}{\partial t^2} + Ay - f, \quad y(x,0) - y_1(x), \quad \frac{\partial y(x,0)}{\partial t} - y_2(x), \\ & \frac{\partial y}{\partial \eta_A} - \int_a^b y(x,t-h)\,dh - Gv, \quad y(x,t') - \Psi_0(x,t') \Bigg) \end{aligned} \quad (30)$$

The operator P is the mapping from the space

$\mathcal{D}^{-1}_{A+\mathcal{D}_t^2}(Q) \times L^2(\Sigma)$ into the space

$\Xi^{-3,-3}(Q) \times \Xi^{-3/2}(\Omega) \times \Xi^{-5/2}(\Omega) \times H^{-5/2}\Xi^{-5/2}(\Sigma) \times H^{-5/2}\Xi^{-5/2}(\Sigma_0)$.

The Fréchet differential of the operator P can be written in the following form:

$$\begin{aligned} P'(y^0,v^0)(\overline{y},\overline{v}) = \Bigg(& \frac{\partial^2 \overline{y}}{\partial t^2} + A\overline{y}, \quad \overline{y}(x,0), \quad \frac{\partial \overline{y}(x,0)}{\partial t}, \\ & \frac{\partial \overline{y}}{\partial \eta_A} - \int_a^b \overline{y}(x,t-h)\,dh - G\overline{v}, \quad \overline{y}\big|_{\Sigma_0}(x,t') \Bigg) \end{aligned} \quad (31)$$

Really, $\dfrac{\partial^2}{\partial t^2}$ (Theorem 2.8 [18]), $A(t)$ (Theorem 2.1 [16]) and $\dfrac{\partial}{\partial \eta_A}$ (Theorem 2.3 [17]) are linear and bounded mappings. Using Theorem 1 [11], we can prove that P' is the operator "one to one" from the space $\mathcal{D}^{-1}_{A+\mathcal{D}_t^2}(Q) \times L^2(\Sigma)$ onto the space $\Xi^{-3,-3}(Q) \times \Xi^{-3/2}(\Omega) \times \Xi^{-5/2}(\Omega) \times H^{-5/2}\Xi^{-5/2}(\Sigma) \times H^{-5/2}\Xi^{-5/2}(\Sigma_0)$.

Considering that the assumptions of the Lusternik's theorem are fulfilled, we can write down the regular tangent cone for the set Q_1 in a point (y^0,v^0) in the form

$$RTC\left(Q_1,(y^0,v^0)\right) = \left((\overline{y},\overline{v}) \in E, \quad P'(y^0,v^0)(\overline{y},\overline{v}) = 0\right) \quad (32)$$

It is easy to notice that it is a subspace. Therefore, using Theorem 10.1 [2] we know the form of the functional belonging to the adjoint cone

$$f_1(\overline{y},\overline{v}) = 0 \qquad \forall (\overline{y},\overline{v}) \in RTC\left(Q_1,(y^0,v^0)\right) \quad (33)$$

b) Analysis of the constraint on controls

The set $Q_2 = Y \times U_{ad}$ representing the inequality constraints is a closed and convex one with non-empty interior in the space E.

Using Theorem 10.5 [2] we find the functional belonging to the adjoint regular admissible cone, i.e.

$$f_2(\overline{y},\overline{v}) \in \left[RAC\left(Q_2,(y^0,v^0)\right)\right]^*$$

We can note if E_1, E_2 are two linear topological spaces, then the adjoint space to $E = E_1 \times E_2$ has the form

$$E^* = \left\{ f = (f_1,f_2); \quad f_1 \in E_1^*,\ f_2 \in E_2^* \right\}$$

and

$$f(x) = f_1(x_1) + f_2(x_2)$$

So we note the functional $f_2(\overline{y},\overline{v})$ as follows

$$f_2(\overline{y},\overline{v}) = f_1'(\overline{y}) + f_2'(\overline{v}) \quad (34)$$

where:
$f_1'(\overline{y}) = 0 \quad \forall y \in Y$ (Theorem 10.1 [2])
$f_2'(\overline{v})$ is a support functional to the set U_{ad} in a point v_0 (Theorem 10.5 [2]).

c) Analysis of the performance functional

Using Theorem 7.5 [2] we find the regular improvement cone of the performance functional (14)

$$RFC\left(I,(y^0,v^0)\right) = \left\{ (\overline{y},\overline{v}) \in E, \quad I'(y^0,v^0)(\overline{y},\overline{v}) < 0 \right\} \quad (35)$$

where: $I'(y^0,v^0)(\overline{y},\overline{v})$ is the Fréchet differential of the performance functional (14) and it can be written as

$$I'(y^0,v^0)(\overline{y},\overline{v}) = 2\lambda_0\lambda_1 \left\langle y^0 - z_d, \overline{y} \right\rangle_{H^{-1,-2}(Q)} + 2\lambda_0\lambda_2 \left\langle Nv^0, \overline{v} \right\rangle_{L^2(\Sigma)} \quad (36)$$

On the basis of Theorem 10.2 [2] we find the functional belonging to the adjoint regular improvement cone, which has the form

$$f_3(\overline{y},\overline{v}) = -\lambda_0\lambda_1 \left\langle y^0 - z_d, \overline{y} \right\rangle_{H^{-1,-2}(Q)} - \lambda_0\lambda_2 \left\langle Nv^0, \overline{v} \right\rangle_{L^2(\Sigma)} \quad (37)$$

where: $\lambda_0 > 0$.

d) Analysis of Euler-Lagrange's equation

The Euler-Lagrange's equation for our optimization problem has the form

$$\sum_{i=1}^{3} f_i = 0 \quad (38)$$

Let $p(x, t)$ be the solution of (21)–(26) for (y^0, v^0). Then, $p(v)$ is defined by transposition, i.e.

$$\langle p, y'' + Ay\rangle = M(y), \qquad \forall y \in \mathcal{D}^{-1}_{A+\mathcal{D}_t^2}(Q) \tag{39}$$

where

$$M(y) = \langle p'' + Ap, y\rangle - \langle p, q\rangle - \langle p(0), y_2\rangle + \langle p'(0), y_1\rangle$$

and y satisfies (1)–(5).

We observe that, for given z_d and v, equations (21)–(26) can be solved backward in time starting from $t = T$, i.e. first solving problem (21)–(26) in the subcylinder Q_1, and in turn in Q_{k-1} etc., until the procedure covers the whole cylinder Q. For this purpose, we may apply Theorem 1.

Lemma 1 *Let the hypothesis of Theorem 1 be satisfied. Then, for given* $z_d \in H^{-1,-2}(Q)$, *and any* $v \in L^2(\Sigma)$, *there exists a unique solution*

$$p(v) \in H^{3,3}(Q) \subset \Xi^{3,3}(Q)$$

to the problem (21)–(26) *defined by transposition* (39).

Next we denote by $\overline{y}$ the solution of $P'(\overline{y}, \overline{v}) = 0$ for any fixed $\overline{v}$. Then taking into account (33)–(34) and (37) we can express (38) in the form

$$f_2'(\overline{v}) = \lambda_0\lambda_1\langle \Lambda_1(y^0 - z_d), \overline{y}\rangle_{H^{-1,-2}(Q)} + \lambda_0\lambda_2\langle Nv^0, \overline{v}\rangle_{L^2(\Sigma)}$$
$$\forall (\overline{y}, \overline{v}) \in RTC(Q_1, (y^0, v^0)). \tag{40}$$

We transform the first component of the right-hand side of (40) using the formulae (21)–(26). Then taking the scalar product of both sides of (21) by an element $\overline{y}(v)$ respectively, we get

$$\lambda_1\langle \Lambda_1(y^0 - z_d), \overline{y}\rangle_{H^{-1,-2}(Q)} = \left\langle \frac{\partial^2 p}{\partial t^2} + A(t)p, \overline{y}\right\rangle_{H^{-1,-2}(Q)} =$$
$$= \left\langle p, \frac{\partial^2 \overline{y}}{\partial t^2}\right\rangle_{H^{-3,-3}(Q)} + \langle A(t)p, \overline{y}\rangle_{H^{-1,-2}(Q)} \tag{41}$$

By using the equation (1), the first term on the right-hand side of (41) can be rewritten as

$$\left\langle p, \frac{\partial^2 \overline{y}}{\partial t^2}\right\rangle_{H^{-3,-3}(Q)} = -\langle p, A(t)\overline{y}\rangle_{H^{-3,-3}(Q)} \tag{42}$$

The second component on the right-hand side of (41) in view of Green's formula can be expressed as

$$\langle A(t)p, \overline{y}\rangle_{H^{-1,-2}(Q)} = \langle p, A(t)\overline{y}\rangle_{H^{-3,-3}(Q)} +$$
$$+ \left\langle p, \frac{\partial \overline{y}}{\partial \eta_A}\right\rangle_{H^{-5/2}\Xi^{-5/2}(\Sigma)} - \left\langle \frac{\partial p}{\partial \eta_A}, \overline{y}\right\rangle_{H^{-5/2}\Xi^{-5/2}(\Sigma)} \tag{43}$$

By using the boundary condition (4), the second term on the right-hand side of (43) can be written as

$$\left\langle p, \frac{\partial \overline{y}}{\partial \eta_A}\right\rangle_{H^{-5/2}\Xi^{-5/2}(\Sigma)} =$$
$$= \left\langle p, \int_a^b \overline{y}(x, t-h)\,dh\right\rangle_{H^{-5/2}\Xi^{-5/2}(\Sigma)} + \langle p, G\overline{v}\rangle_{H^{-5/2}\Xi^{-5/2}(\Sigma)} =$$
$$= \left\langle \int_a^b p(x, t'+h)\,dh, \overline{y}(x, t')\right\rangle_{H^{-5/2}\Xi^{-5/2}[\Gamma\times(-h, T-b)]} +$$
$$+ \left\langle \int_a^{T-t} p(x, t'+h)\,dh, \overline{y}(x, t')\right\rangle_{H^{-5/2}\Xi^{-5/2}[\Gamma\times(T-b, T-a)]} = \langle p, G\overline{v}\rangle_{H^{-5/2}\Xi^{-5/2}(\Sigma)} \tag{44}$$

The last component in (43) may be written as

$$\left\langle \frac{\partial p}{\partial \eta_A}, \overline{y}\right\rangle_{H^{-5/2}\Xi^{-5/2}(\Sigma)} = \left\langle \frac{\partial p}{\partial \eta_A}, \overline{y}\right\rangle_{H^{-5/2}\Xi^{-5/2}[\Gamma\times(0, T-b)]} +$$
$$+ \left\langle \frac{\partial p}{\partial \eta_A}, \overline{y}\right\rangle_{H^{-5/2}\Xi^{-5/2}[\Gamma\times(T-b, T-a)]} + \left\langle \frac{\partial p}{\partial \eta_A}, \overline{y}\right\rangle_{H^{-5/2}\Xi^{-5/2}[\Gamma\times(T-a, T)]} \tag{45}$$

Substituting (44) and (45) into (43) and then (42) and (43) into (41) we obtain

$$\lambda_1\langle \Lambda_1(y^0 - z_d), \overline{y}\rangle_{H^{-1,-2}(Q)} = -\langle p, A(t)\overline{y}\rangle_{H^{-3,-3}(Q)} + \langle p, A(t)\overline{y}\rangle_{H^{-3,-3}(Q)} +$$
$$+ \left\langle \int_a^b p(x, t+h)\,dh, \overline{y}\right\rangle_{H^{-5/2}\Xi^{-5/2}[\Gamma\times(-h, 0)]} + \left\langle \int_a^b p(x, t+h)\,dh, \overline{y}\right\rangle_{H^{-5/2}\Xi^{-5/2}[\Gamma\times(0, T-b)]} +$$
$$+ \left\langle \int_a^{T-t} p(x, t+h)\,dh, \overline{y}\right\rangle_{H^{-5/2}\Xi^{-5/2}[\Gamma\times(T-b, T-a)]} + \langle p, G\overline{v}\rangle_{H^{-5/2}\Xi^{-5/2}(\Sigma)} -$$
$$- \left\langle \frac{\partial p}{\partial \eta_A}, \overline{y}\right\rangle_{H^{-5/2}\Xi^{-5/2}[\Gamma\times(0, T-b)]} - \left\langle \frac{\partial p}{\partial \eta_A}, \overline{y}\right\rangle_{H^{-5/2}\Xi^{-5/2}[\Gamma\times(T-b, T-a)]} -$$
$$- \left\langle \frac{\partial p}{\partial \eta_A}, \overline{y}\right\rangle_{H^{-5/2}\Xi^{-5/2}[\Gamma\times(T-a, T)]} = \langle p, G\overline{v}\rangle_{H^{-5/2}\Xi^{-5/2}(\Sigma)} = \langle G^* p, \overline{v}\rangle_{L^2(\Sigma)} \tag{46}$$

Substituting (46) into (40) gives

$$f_2'(\overline{v}) = \lambda_0\langle G^* p + \lambda_2 N v^0, \overline{v}\rangle_{L^2(\Sigma)} \tag{47}$$

Using the definition of the support functional [2] and dividing both sides of the obtained inequality by λ_0, we finally get

$$\langle G^* p(v^0) + \lambda_2 N v^0, v - v^0\rangle_{L^2(\Sigma)} \geq 0 \qquad \forall v \in U_{ad} \tag{48}$$

The last inequality is equivalent to the maximum condition (27).

The uniqueness of the optimal control follows from the strict convexity of the performance functional (14).

This last remark finishes the proof of Theorem 2.

One may also consider analogous optimal control problem with the performance functional

$$\hat{I}(y,v) = \lambda_1 \left\| y(v)|_{\Sigma} - z_{\Sigma d} \right\|^2_{H^{-5/2}\Xi^{-5/2}(\Sigma)} + \lambda_2 \left\langle (Nv), v \right\rangle_{L^2(\Sigma)} \quad (49)$$

where: $z_{\Sigma d}$ is a given element in $H^{-5/2}\Xi^{-5/2}(\Sigma)$; we assume that the space $H^{-5/2}\Xi^{-5/2}(\Sigma)$ is such that $y(v)\big|_{\Sigma} \in H^{-5/2}\Xi^{-5/2}(\Sigma)$. Then the solution of the formulated optimal control problem is equivalent to seeking a pair

$$\left\langle y^0, v^0 \right\rangle \in E = \mathcal{D}^{-1}_{A+\mathcal{D}_t^2}(Q) \times L^2\left(\Sigma\right)$$

that satisfies the equation (1)–(5) and minimizing the cost function (49) with the constraints on control (15).

We can prove the following theorem:

Theorem 3 *The solution of the optimization problems* (1)–(5), (49), (15) *exists and it is unique with the assumptions mentioned above; the necessary and sufficient conditions of the optimality are characterized by the following system of partial differential equations and inequalities:*

State equation (1)–(5),

Adjoint equations

$$\frac{\partial^2 p}{\partial t^2} + A\left(t\right)p = 0 \qquad x \in \Omega,\ t \in \left(0,T\right) \quad (50)$$

$$\frac{\partial p}{\partial \eta_A} = \int_a^b p\left(x,t+h\right)dh + \lambda_1 \Lambda_2 \left(y^0|_{\Sigma} - z_{\Sigma d}\right) \qquad x \in \Gamma,\ t \in \left(0,T-b\right) \quad (51)$$

$$\frac{\partial p}{\partial \eta_A} = \int_a^{T-t} p\left(x,t+h\right)dh + \lambda_1 \Lambda_2 \left(y^0|_{\Sigma} - z_{\Sigma d}\right) \qquad x \in \Gamma,\ t \in \left(T-b,T-a\right) \quad (52)$$

$$\frac{\partial p}{\partial \eta_A} = \lambda_1 \Lambda_2 \left(y^0\ |_{\Sigma} - z_{\Sigma d}\right) \qquad x \in \Gamma,\ t \in \left(T-a,T\right) \quad (53)$$

$$p\left(x,T\right) = 0 \qquad z \in \Omega \quad (54)$$

$$\frac{\partial p\left(x,T\right)}{\partial t} = 0 \qquad x \in \Omega \quad (55)$$

where: Λ_2 is a canonical isomorphism of $H^{-5/2}\Xi^{-5/2}(\Sigma)$ into $H^{5/2}\Xi^{5/2}(\Sigma)$.

Maximum condition

$$\left\langle G^* p\left(v^0\right) + \lambda_2 N v^0, v - v^0 \right\rangle_{L^2(\Sigma)} \geq 0 \qquad \forall v \in U_{ad} \quad (56)$$

Moreover, it can be proved the following result.

Lemma 2 *Let the hypothesis of Theorem 1 be satisfied. Then, for given* $z_{\Sigma d} \in H^{-5/2}\Xi^{-5/2}(\Sigma)$ *and any* $v \in L^2(\Sigma)$, *there exists a unique solution* $p(v) \in H^{3,3}(Q) \subset \Xi^{3,3}(Q)$ *to the problem* (50)–(55) *defined by transposition* (39).

The idea of the proof of the Theorem 3 is the same as in the case of the Theorem 2.

In the case of performance functionals (14) and (49) with $\lambda_1 > 0$ and $\lambda_2 = 0$, the optimal control problem reduces to the minimizing of the functional on a closed and convex subset in a Hilbert space. Then, the optimization problem is equivalent to a quadratic programming one ([10, 11] which can be solved by the use of the well-known algorithms, e.g. Gilbert's [1, 10, 11]).

4. Conclusions and Perspectives

The derived conditions of the optimality (Theorems 2 and 3) are original from the point of view of application of the Dubovicki-Milyutin theorem [6] in solving optimal boundary control problems for second order hyperbolic systems in which integral time lags appear in the Neumann boundary conditions.

The existence and uniqueness of solutions for such hyperbolic systems are presented – Theorem 1. The optimal control is characterized by using the adjoint equations – Lemmas 1 and 2. Necessary and sufficient conditions of optimality with the quadratic performance functionals (14) and (49) and constrained control (15) are derived for the Neumann problem (Theorems 2 and 3).

The proved optimization results (Theorems 2 and 3) constitute a novelty of the paper with respect to the reference [11] concerning application of the Lions scheme [16] for solving linear quadratic hyperbolic problems of optimal control.

The proposed methodology based on the Dubovicki-Milyutin scheme can be presented as a specific case study concerning hyperbolic problems described by partial differential equations of the hyperbolic type including time lags appeared in the integral form for the case $h \in \left(0,b\right)$ both in the state equations and in the Neumann boundary conditions.

Another direction of research will be numerical examples concerning the determination of optimal control with constraints for integral time delay hyperbolic systems.

Appendix

Apart from lumped delays, which lead to difference-differential equations, control systems may incorporate so-called distributed delays. These delays occur in distributed parameter systems represented by partial differential equations. The majority of thermal processes, together with processes in which the signal is transmitted by long electric, hydraulic or pneumatic lines, show a delay distributed along the entire length of spatial coordinate. This time delay is usually accompanied by disturbances introduced to the transmitted signal. Processes of this type are often described by partial differential equations.

Distributed time delays constitute a particular case of integral time delays. Such problems concerning integral time delays have not been investigated sufficiently well till now. Consequently, the Author solved an abstract optimal boundary control problem for hyperbolic systems with boundary conditions involving integral time lags.

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Problemy ekstremalne dla systemów hiperbolicznych z warunkami brzegowymi, w których występują całkowe opóźnienia czasowe

Streszczenie: Zaprezentowano ekstremalne problemy dla systemów hiperbolicznych z całkowymi opóźnieniami czasowymi. Rozwiązano problem optymalnego sterowania brzegowego dla systemów hiperbolicznych drugiego rzędu, w których całkowe opóźnienia czasowe występują w warunkach brzegowych typu Neumanna. Tego rodzaju systemy stanowią w liniowym przybliżeniu uniwersalny model matematyczny procesów fizycznych, w których ma miejsce przesyłanie sygnałów na odległość w liniach długich typu elektrycznego, hydraulicznego i innych. Korzystając ze schematu Dubowickiego-Milutina wyprowadzono warunki konieczne i wystarczające optymalności dla problemu liniowo-kwadratowego.

Słowa kluczowe: sterowanie brzegowe, systemy hiperboliczne, warunki brzegowe typu Neumanna, całkowe opóźnienia czasowe

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